

A locus problem

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Here is a meaningful locus problem sent to me by Ken Ho :

Moving points B and C are on circle $x^2 + y^2 = 25$. Let A be (2, 0) and M be mid-point of BC. If $\triangle BAC$ is a straight line, what is the locus of M?

The main difficulty of this question is that there are 3 movable points, B, C and M. The chief movable point is M. B and C are "guests" and should leave the house later.

Method 1

Let $M(x, y)$, $B(x_1, y_1)$ and $C(x_2, y_2)$.

Let $y = m(x - 2)$ be the line BC, where m is the slope of the line. Note that this line passes through $A(2, 0)$.

Substitute this line into the circle $x^2 + y^2 = 25$ to find the intersection points. We get :

$$\begin{aligned}x^2 + [m(x - 2)]^2 &= 25. \\x^2 + m^2[x^2 - 4x + 4] &= 25 \\(1 + m^2)x^2 - 4m^2x + (4m^2 - 25) &= 0\end{aligned}$$

Now, this is a quadratic equation with roots x_1, x_2 .

The sum of roots $= x_1 + x_2 = \frac{4m^2}{1+m^2}$.

Since M is the mid point of B, C we have $(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$\therefore x = \frac{x_1+x_2}{2} = \frac{1}{2} \frac{4m^2}{1+m^2} = \frac{2m^2}{1+m^2} \quad \dots (1)$$

We still need to find y .

Since $B(x_1, y_1)$ and $C(x_2, y_2)$ are on the straight lines $y = m(x - 2)$, we have

$$y_1 = m(x_1 - 2) \quad \dots (2)$$

$$y_2 = m(x_2 - 2) \quad \dots (3)$$

$$(2) + (3), \quad y_1 + y_2 = m(x_1 + x_2) - 4m$$

$$\therefore y = \frac{y_1+y_2}{2} = m\left(\frac{x_1+x_2}{2}\right) - 2m = m\frac{2m^2}{1+m^2} - 2m = \frac{2m^3 - 2m(1+m^2)}{1+m^2} = -\frac{2m}{1+m^2} \quad \dots (4)$$

We now get x and y . Our aim is to find a relation between x and y , that is, we need to eliminate m in (1) and (4).

$$\frac{(1)}{(4)}, \quad \frac{x}{y} = \frac{\frac{2m^2}{1+m^2}}{-\frac{2m}{1+m^2}} = -m \quad \dots (5)$$

$$(5) \downarrow (1), \quad x = \frac{2m^2}{1+m^2} = \frac{2\left(\frac{x}{y}\right)^2}{1+\left(\frac{x}{y}\right)^2} = \frac{2x^2}{x^2+y^2} \Rightarrow 1 = \frac{2x}{x^2+y^2}$$

$$x^2 + y^2 - 2x = 0$$

Therefore the locus of M is $x^2 + y^2 - 2x = 0$,

which is a circle with centre (1, 0) and radius 1. This circle also passes through A(2, 0) and the origin (0, 0).

Method 2

This question can be easily done using polar co-ordinates. Let us revise a little bit on polar co-ordinate system. The relations between Cartesian and polar system are given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$

As an example the given circle $x^2 + y^2 = 25$ can be re-written simply as $r = 5$.

Let $A(2, 0^\circ)$, $M(r, \theta)$ and the pole be O . Note that both A and M are in polar-form.

Let $P(R, \Theta)$ be a variable point on the line BC.

Here we use the variables R, Θ so as not to mix up with r, θ of point M.

Since M is the mid-point of BC and $OM \perp BC$, OM is the normal of the line BC!

As $M=(r, \theta)$ the normal form of the line BC is therefore given by:

$x \cos \theta + y \sin \theta = r$, where r is the length of normal and θ angle of inclination of normal.

Change this equation to polar form, we get:

$$(R\cos\Theta)\cos\theta + (R\sin\Theta)\sin\theta = r$$

$$R\cos(\Theta - \theta) = r$$

As this straight line passes through the point $A(2, 0^\circ)$, we have

$$\therefore 2\cos(0^\circ - \theta) = r$$

$\therefore r = 2\cos\theta$, which is already an equation for the locus of M in polar form!

$r = 2\cos\theta$ is an equation of a circle. You can change back to Cartesian co-ordinate form.

Multiply both sides by r , we get:

$$r^2 = 2(r\cos\theta)$$

$$x^2 + y^2 = 2x$$

Therefore the locus of M is $x^2 + y^2 - 2x = 0$.

The main interesting point is that, in the above polar calculation, the workings do not depend on the given circle. You can therefore replace the given circle by any circle, provided that the centre is at the origin and A(2, 0) is inside the circle. You can replace the given circle by, say, $x^2 + y^2 = 100$, you get the same locus of M: $x^2 + y^2 - 2x = 0$!